Mathematical Programming Problems

FAROOQ AHMAD SHAH
Research Scholar, Glocal University,
Saharanpur, U.P.

DR. MOHAMMAD SHADAB
Assistant Professor, Glocal University,
Saharanpur, U.P.

Abstract:
We present a brief survey of related work done in the fields of multi-objective mathematical programming, optical control. Many problems of practical importance can be transformed into different forms of minimization or maximization problems no matter whether such problems are from the field of engineering, science, business or finance. These problems share the solution that offers certain optimal criteria under several limitations. Finally, one can say that nothing at all takes place in the universe in which some rule of the maximum or minimum does not appear. Kuhn and Tucker were the first to incorporate some interesting results concerning multi-objective optimization in 1951. Since then, research in this area has made remarkable progress both theoretically and practically. Duality in non linear programming problems originated with duality results of quadratic programming, initially studied by Dannis. Dual of convex primal program was given by Dorn and Mangasarian. Concept of mixed type multi-objective duality seems to be quite interesting and useful from practical point as well as algorithm point of view. The computational advantage of mixed type dual formalities involves the flexibility of the choice of constraints to be put in the Lagrange function can be exploited to develop certain efficient solution procedure for solving mathematical programing problems.

1. Introduction

Mathematical programming earned a status of scientific field in its own right during late 1940s and since then it has undergone significant development. It is now regarded as one of the most vital and exciting part of modern mathematics having applications in various scientific disciplines such as, engineering economics and natural sciences.

A general mathematical programming problem (MPP) can be stated as:

(MP) Optimize (minimize/maximize) f(x)

Subject to

g_i (x) ≤0 (i = 1,2,….,m)

h_j (x) = (j=1,2,…..,k), x∈X

where (i) x = $(x_1, x_2, x_3,\ldots\ldots,x_n)^T$ is the vector of unknown variables and (ii) f, g_i (i = 1,2,……., m), h_j (j = 1,2,…..,k) are the real valued functions of real n variables x_1, x_2, …., x_n and $X \subseteq R^n$. in this formulation, the function f is called the objective function. The constraints, gi ≤(x) 0, i = 1,2,…,m are referred to as an inequality constraints. h_j (x)= 0, j= 1,2,3,…,k are called equality constraints. the inclusion x∈X is known as abstract constraints. Optimality conditions and duality have played a vital role in the progress of mathematical programming. Fritz John was the first to derive necessary optimality conditions for constrained optimization problem using a Long range multiplier rule. The principle of duality connects two programs, one of which is called the primal problem and the other is called the dual problem, in such a way that the existence of an optimal solution to one of them guarantees an optimal solution to other. Multi-objective optimization is the art of detecting and making good compromises. It is based upon the fact that most real-world decisions are compromises between partially conflicting objectives that cannot easily be offset against each other. Thus, one is forced to look for possible compromises and finally decide which one to implement. So, the final decision in multi-objective optimization is Mond and Hanson formulated the following pair of dual variation problem:

Primal problem:

$$\text{Minimize } \int_I f(t,x,\dot{x})\,dt$$

Subject to

$$x(a)=\alpha,\quad x(b)=\beta,$$

$$g(t,x,\dot{x})\le 0,\qquad t\in I.$$

Dual problem:

$$\text{Maximize } \int_a^b \left\{ f(t,u(t),\dot{u}(t)) + y(t)^T g(t,u(t),\dot{u}(t)) \right\} dt$$

Subject to

$$u(a)=\alpha,\quad u(b)=\beta,$$

$$f_u(t,u(t),\dot{u}(t)) + y(t)^T g_u(t,u(t),\dot{u}(t))$$
$$-D\left[f_{\dot{u}}(t,u(t),\dot{u}(t)) + y(t)^T g_{\dot{u}}(t,u(t),\dot{u}(t)) \right]=0,\quad t\in I$$

$$y(t)\ge 0,\qquad t\in I$$

Where

i)I=[a,b], a real interval and

ii)F: I x R^n x R^n → R, g: I x R^n x R^n → R^m are continuously differentiable and y: I→ R^m is piecewise smooth functions.

Let θ be a numerical function defined on an open set Γ in R^n, then $\nabla f(\bar{x})$ denotes the gradient of θ at $\bar{x}$, that is

$$\nabla f(\bar{x}) = \left[\frac{\partial f(\bar{x})}{\partial x^1}, \ldots, \frac{\partial f(\bar{x})}{\partial x^2} \right]^T$$

Let φ be a real valued twice continuously differentiable function defined on an open set contained in R^n x R^m. Then $\nabla_x \varphi$ (x,y) and $\nabla_y \varphi$ (x,y) denote the gradient (column) vector of φ with respect to x and y respectively.

i.e.,

$$\nabla_x \phi(\bar{x},\bar{y}) = \left(\frac{\partial \phi}{\partial x^1}, \frac{\partial \phi}{\partial x^2}, \ldots, \frac{\partial \phi}{\partial x^n} \right)^T_{(\bar{x},\bar{y})}$$

$$\nabla_y \phi(\bar{x},\bar{y}) = \left(\frac{\partial \phi}{\partial y^1}, \frac{\partial \phi}{\partial y^2}, \ldots, \frac{\partial \phi}{\partial y^n} \right)^T_{(\bar{x},\bar{y})}$$

Further, $\nabla^2_{xx}\phi(\bar{x},\bar{y})$ and $\nabla^2_{xy}\phi(\bar{x},\bar{y})$ denote respectively the $(n\times n)$ and $(n\times m)$ matrices of second- order partial derivative i.e.,

$$\nabla^2_{xx}\phi(\bar{x},\bar{y})=\left(\frac{\partial^2\phi}{\partial x^i x^j}\right)_{(\bar{x},\bar{y})}$$

$$\nabla^2_{xy}\phi(\bar{x},\bar{y})=\left(\frac{\partial^2\phi}{\partial x^i x^j}\right)_{(\bar{x},\bar{y})}$$

The symbols $\nabla^2_{yy}\phi(\bar{x},\bar{y})$ and $\nabla^2_{yx}\phi(\bar{x},\bar{y})$ are similarly defined.

Consider the following multiobjective programming problem:

(VP): Minimize $\phi(z)=(\phi_1(x),\phi_2(x),...,\phi_p(x))$

Subject to

$$h_j(x)\le 0,\ j=1,2,...n$$

A feasible point is $\bar{x}$ id to be a weak minimum of (VP), if there does not exist any x€X_0 such that $_\varphi$(x) < $_\varphi$($\bar{x}$). A feasible point $\bar{x}$ is said to be properly efficient solution of (VP) , if it is an efficient solution of (VP) and if there exists $\bar{x}$ scalar M>0 such that $\bar{x}$ or each i and x€x_0 satisfying $_{\varphi i}$(x) < $_{\varphi i}$($\bar{x}$), we have

$$\frac{\phi_i(\bar{x})-\phi_i(x)}{(\phi_j(x)-\phi_j(\bar{x})}\le M,$$

For some j, satisfying $_{\varphi j}$ (x)> $_{\varphi j}$ ($\bar{x}$
An efficient point $\bar{x}$ €X $\bar{x}$ at is not properly efficient is said to be improperly efficient. Then $\bar{x}$ is improperly efficient means that every scale M>0 (no matter how large), then point x€X and i such that Ψ^i (x) < Ψ^i ($\bar{x}$)
and

$$\frac{\psi^i(\bar{x})-\psi^i(x)}{\psi^j(x)-\psi^j(\bar{x})}>M,$$

for all j satisfying $\psi^j(x)>\psi^j(\bar{x})$

2. Review of literature
Consider the non-linear programming problem:
(P): Minimize f(x)
Subject to
$h_j(x) \le 0$, (j=1,2,….,m)
Where f: $R^n \to R$ and $h_j : R^n \to R$, (j=1,2,….,m) are continuously differentiable. The following problem:
(WD) : Maximize f(x) + y^T h(x)
Subject to
$\nabla(f(x) + y^T h(x)) = 0$
y≥ 0, y€R^m

is known as the Wolfe type dual for the problem (P), Mangasarian explained by means of an example that certain duality theorems may not be valid if the objective or constraint function is a generalized convex function. This motivated Mond and Weir to introduce a different dual for(P) as

(MWD) : Maximize f(x)

Subject to

$\nabla f(x) + \nabla y^T h(x)=0$

$y^T h(x) \geq 0$

$y\geq 0,\ y\in R^m$

and they proved various duality theorems under pseudo convexity of f and quasiconvexity of $y^T h(.)$ for all feasible solution of (P) and (MWD).

Mond considered the following class of non-differentiable mathematical programming problems:

(NP): **Minimize** $f(x)+\left(x^T Bx\right)^{\frac{1}{2}}$

Subject to

$$h_j(x)\leq 0, \quad j=1,2,\ldots,m,$$

where f and h_j , $j=1,2,\ldots,m$ are twice differentiable function from R^n to R and B is an n x n positive semidefinite (symmetric) matrix. It is assumed that the functions f and h_j , $j=1,2,\ldots,m$ are convex functions. They established a duality theorem between (NP) and the following problem

(ND): **Maximize** $f(u)+y^T h(u)-u^T\nabla\left[f(u)+y^T h(u)\right]$

Subject to

$$\nabla f(u)+\nabla y^T h(u)+Bw=0,$$

$$w^T Bw\leq 1$$

$$y\geq 0.$$

Further on the lines of Mond and Weir , Chandra, Craven and Mond introduced another dual program:

(NWD): **Maximize** $f(u)-u^T\nabla\left[f(u)+y^T h(u)\right]$

Subject to

$$\nabla f(u)+\nabla y^T h(u)+Bw=0,$$

$$y^T h(u)\geq 0,$$

$$w^T Bw\leq 1,$$

$$y\geq 0.$$

and established duality theorems by assuming the function $f(\cdot)+(\cdot)^T Bw$ to be pseudoconvex and $y^T h(\cdot)$ to be quasiconvex for all feasible solutions of (NP) and (NWD).

Later, Mond and Schechter replaced the square root term by the norm term and considered the nondifferentiable nonlinear programming problems as:

$(NP)_1$: Minimize $f(x)+\|s_x\|_p$

Subject to

$h_j(x)\leq 0, j=1,2\ldots m$

Here f and h_j ,(j=1,2,….m) are twice differentiable function from R^n to R. the dual for $(NP)_1$ is the problem:

$(ND)_1$: maximum $f(u)=y^T h(u)-U^T s^T v$

Subject to

$$\nabla f(u)+\nabla y^T h(u)+S^T v=0,$$

$\|v\|_q\leq 1$,

$y\geq 0$,

Where p and q are conjugate exponents.

Later Schechter replaced the norm term or the square rot term by a more general function as the support function of a compact set. The problem considered by Schechter is:

$(NP)_2$: minimize $f(x)=S(x/_C)$

Subject to

$h_j(x)\leq 0, j=1,2\ldots,m$

Where f and h_j(j=1,2,….,m) are twice differentiable function from R^n to R and $S(x/_C)$ is a support fuction of a compact convex set $C\subseteq R^n$. Using the sub differential of the support function of $S(x/_C)$, the dual of $(NP)_2$ is the problem:

$(ND)_2$: Maximize $f(u)+w^T u+y^T h(u)$

Subject to

$$\nabla f(u)+\nabla y^T h(u)+w=0,$$

$$y\geq 0,\ w\in C.$$

3. Research methodology

The present study is based on historical, explanatory and descriptive. The proposed examination has utilized primary and secondary data for examination. It has depended on the narrative investigation of accessible essential source material. The books, investigate articles and other distibuted materials comprise the secondary data of the examination.

4. Objectives of the study

1. To study duality and mixed type duality for control problems.
2. To study multi objective duality in mathematical programming problems.
3. To study second order duality for variational problems.

5. Analysis and Results

We consider the following non-differentiable non-linear problem with support functions:

(NP): Min $f(x)+S(x/_C)$

Subject to

$$g_j(x)+S(x/_{Dj})\leq 0,\ (j=1,2\ldots,m)$$

For this problem, we construct the following Wolfe and Mond Weir type second order dual.

$$\text{(WD): Max } f(u)+z^T u+\sum_{j=1}^{m} y_j\left(g_j(u)+w_j^T(u)\right)-\tfrac{1}{2} p^T\nabla^2\left(f(u)+y^T g(u)\right)p$$

Subject to,

$$\nabla(f(u)+z^T u)+\sum_{j=1}^{m} y_j\nabla\left(g_j(u)+w_j\right)+\nabla^2\left(f(u)+y^T g(u)\right)p=0,$$

$$y\geq 0,$$

$$z\in C, w_j\in D_j,\ \ (j=1,2.,,,m).$$

Mond-Weir type second-order dual for the problem (NP).

$$\text{(SM-WD): Max } f(u)+z^T u-\tfrac{1}{2} p^T\nabla^2\left(f(u)\right)p$$

Subject to,

$$\nabla f(u)+z+\sum_{j=1}^{m} y_j\left(\nabla g_j(u)+w_j\right)+\nabla^2\left(f(u)+y^T g(u)\right)p=0,$$

$$\sum_{j=1}^{m} y_j\left(g_j(u)+w_j^T u\right)-\tfrac{1}{2} p^T\nabla^2\left(y^T g(u)\right)p\geq 0,$$

$$y\geq 0,$$

$$z\in C, w_j\in D,\quad \forall j=1,2,\ldots m$$

for the pair of Wolfe type second-order dual problem (NP) and (WD) usual duality theorems are validated under second order convexity, and for the pair of second order Mond-Weir problem (NP) and (M-WD) , various duality theorems are validated under second order generalized convexity. Special cases are also deduced.

Following pair of second order symmetric dual programs with cone constraint is formulated:

$$\text{(SP): Minimize } G(x,y,p)=f(x,y)-y^T(\nabla_y f(x,y)+\nabla_y^2 f(x,y)p)$$
$$-\frac{1}{2}p^T\nabla_y^2 f(x,y)p$$

Subject to,

$$-\nabla_y f(x,y)-\nabla_y^2 f(x,y)p\in C_2^*$$

$$(x,y)\in C_1\times C_2$$

$$\text{(SD): Maximize } H(x,y,q)=f(x,y)-x^T(\nabla_x f(x,y)+\nabla_x^2 f(x,y)q)$$
$$-\frac{1}{2}q^T\nabla_x^2 f(x,y)q$$

Subject to,

$$\nabla_x f(x,y)+\nabla_x^2 f(x,y)q\in C_1^*$$

$$(x,y)\in C_1\times C_2$$

Where (i) f: $c_1 \times c_2 \rightarrow R$ is a twice differentiable function,
(ii): c_1 and c_2 are closed convex cones with nonempty interior in R^n and R^m respectively.
(iii) c_1^* and c_2^* are positive polar cones of c_1 and c_2 respectively.
For this pair of problems various duality theorems including self-duality theorems are proved under second order convexity.

Following pair of second order mixed integer symmetric and self-duality is investigated.
Primal problem:

$$\text{(MSP)}: \quad \underset{x^1}{\text{Max}}\ \underset{x^2,y,s}{\text{Min}}\ \phi(x,y,s) = f(x,y) - (y^2)^T\left(\nabla_{y^2} f(x,y) + \nabla^2_{y^2} f(x,y)s\right) - \frac{1}{2}s^T\nabla^2_{y^2}\ f(x,y)s$$

Subject to,

$$-\nabla_{y^2} f(x,y) - \nabla^2_{y^2} f(x,y)s \in K_2^*$$

$$x^1 \in U, (x^2, y) \in K_1 \times T.$$

and

Dual Problem

$$\text{(MSD)}: \quad \underset{y^1}{Min}\underset{x,y^2,r}{Max}\,\psi(x,y,r) = f(x,y) - (x^2)^T\left(\nabla_{x^2} f(x,y) + \nabla^2_{x^2} f(x,y)r\right) f(x,y) - \frac{1}{2}(r^T)^T\nabla^2_{x^2} f(x,y)r$$

Subject to,

$$\nabla_{x^2} f(x,y) + \nabla^2_{x^2} f(x,y)r \in K_1^*$$

$$y^1 \in V, (x, y^2) \in S \times K_2$$

where $s \in R^{m-m_1}$ and $r \in R^{n-n_1}$.

6. Conclusion

Following Wolfe type non-differentiable multi objective Second order symmetric dual problems are formulated and for this pair of problem weak, strong and self-duality theorems are established under suitable convexity conditions.

$$\text{Primal (SWP)}: \text{Minimize } F(x,y,z,p) = F_i(x,y,z_1,p),\ldots F_k(x,y,z_k,p)$$

Subject to,

$$\sum_{i=1}^{k}\lambda_i\left(\nabla_2 f_i(x,y) - z_i + \nabla_2^2 f_i(x,y)p\right) \leqq 0$$

$$z_i \in D_i, i = 1,2,\ldots,k$$

$$x \geqq 0$$

$$\lambda \in \wedge^+$$

Wolfe type dual to the problem (SWP) is:

Dual (SWD): **Minimize** $G(u, v, w, q) = G_1(u, v, w_1, q), \ldots G_k(u, v, w_k, q)$

Subject to,

$$\sum_{i=1}^{k} \lambda_i \left(\nabla_1 f_i(u, v) - w_i + \nabla_1^2 f_i(u, v) q \right) \geqq 0$$

$$w_i \in C_i, \; i = 1, 2, \ldots, k$$

$$v \geqq 0$$

$$\lambda \in \wedge^+$$

where

i. $F_i(x, y, z_i, p) = f_i(x, y) + s(x \mid C_i) - y^T z_i - \frac{1}{2} p^T \nabla_2^2 f_i(x, y) p$

$- y^T \sum_{i=1}^{k} \lambda_i \left(\nabla_2 f_i(x, y) - z_i + \nabla_2^2 f_i(x, y) p \right)$

ii. $G_i(u, v, w_i q) = f_i(u, v) - s(v \mid D_i) + u^T w_i - \frac{1}{2} q^T \nabla_1^2 f_i(u, v) q$

$- u^T \sum_{i=1}^{k} \lambda_i \left(\nabla_1 f_i(u, v) - w_i + \nabla_1^2 f_i(u, v) q \right)$, and

iii. For each i, $s(x \mid C_i)$ and $s(v \mid D_i)$ represent support functions of compact convex sets C_i and D_i in R^n and R^m, respectively.

iv. $w = (w_1, \ldots w_K)$ with $w_i \in C_i$ and $z = (z_1, \ldots z_K)$ for each $\{i = 1, 2, \ldots, k\}$

v. $\wedge^+ = \left\{ \lambda \in R^k \mid \lambda = (\lambda_i, \ldots \lambda_k), \lambda > 0, \sum_{i=1}^{k} \lambda_i = 1 \right\}$

Following pair of mixed type multi objective second order symmetric and problems is formulated.
Primal problem:

(SMP): **Minimize** $F(x^1, x^2, y^1, y^2, p, r)$

$$= \left(F_1(x^1, x^2, y^1, y^2, p, r), \ldots, F_k(x^1, x^2, y^1, y^2, p, r) \right)$$

Subject to,

$$\nabla_{y^1} (\lambda^T f)(x^1, y^1) + \nabla_{y^1}^2 (\lambda^T f)(x^1, y^1)\, p \leqq 0,$$

$$\nabla_{y^2}(\lambda^T g)(x^2, y^2) + \nabla^2_{y^2}(\lambda^T g)(x^2, y^2)\, r \leqq 0,$$

$$(y^2)^T\left[\nabla_{y^2}(\lambda^T g)(x^2, y^2) + \nabla^2_{y^2}(\lambda^T g)(x^2, y^2)\, r\right] \geqq 0,$$

$$x^1, x^2 \geqq 0,$$

$$\lambda \in \Lambda^+.$$

Dual Problem:

(SMD): **Max** $G(u^1, u^2, v^1, v^2, q, s) = \left(G_1(u^1, u^2, v^1, v^2, q, s), \dots, G_k(u^1, u^2, v^1, v^2, q, s)\right)$

Subject to,

$$\nabla_{x^1}(\lambda^T f)(u^1, v^1) + \nabla^2_{x^1}(\lambda^T f)(u^1, v^1)\, q \geqq 0,$$

$$\nabla_{x^2}(\lambda^T g)(u^2, v^2) + \nabla^2_{x^2}(\lambda^T g)(u^2, v^2)\, s \geqq 0,$$

$$\left(u^2\right)^T\left[\nabla_{x^2}(\lambda^T g)(u^2, v^2) + \nabla^2_{x^2}(\lambda^T g)(u^2, v^2)\, s\right] \leqq 0,$$

$$v^1, v^2 \geqq 0,$$

$$\lambda \in \Lambda^+.$$

This formulation of the programs not only generalizes mixed type first order symmetric multi objective duality results but also unifies the pair of wolfe and Mond-weir type second-order symmetric multi objective programme

References

1. Abadie, J. (1967). “Non linear programming, North Holland publishing Company, Amsterdam
2. Chandra, S. and Durga Prasad, M.V (1992). “constrained vector value games and multi objective programming”.
3. Dorn,W.S, “ a duality theorem for convex program”
4. Geofferion, A. M, “ Proper efficiency art the theory of vector of maximization”
5. Husain, I and Ahmad,A “Multi objective symetic duality with invexity”.
6. _________. and Jabeen, Z. “on variational problems involving higher order derivatives”
7. Mangasarian, O.L (1969), Non linear programming McGraw Hill, New York.
8. Mond, B and Hansom, M.A, “symmetric duality for variational problems.”
9. _________. and M.A Hanson, duality for control problems.
10. _________. second order duality in non linear programming
11. Schonfeld, P., “some duality theorem for non-linear vector maximum problem.”
12. Singh, C, Duality theorem in multi objective differentiable programming.
13. Valentine, F.A the problem of Lagrange with differentiable in equalities as added side constants, contribution to the calculus of variation

14. XU, mixed type duality in multi objective programming.
15. Zhang, J., and Mond B. Duality for a non differentiable programming problem.